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The Strong Uniqueness Theorem for Monosplines

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DEDICATED TO THE MEMORY OF GÉZA FREUD

1. INTRODUCTION

One of the most beautiful results of Linear Uniform Approximation Theory is Freud's Uniqueness Theorem [17]. His result has been generalized by Newman and Shapiro [14] using the "strong uniqueness approach." In the non-linear case the earliest theorems of this type for rational functions, are due to Maehly and Witzgall [16], and Cheney and Loeb [15]. More recently a more general strong unicity theorem was proven for elements of maximal dimension in a varisolvent family [1]. All of these results depend strongly on the technique of showing that the difference of two supposed best approximations has too many zeros. When one deals with monosplines having knots of multiplicity greater than one [2], this technique fails. In this paper we consider this type of approximation problem for Polynomial and Extended Totally Positive Monosplines and in the process develop an improvement theorem.

2. THE STRONG UNIQUENESS THEOREM

Let P be an open set in R^n and consider a family of functions $F=\{F(A, X): A \in P\}$ such that, for each $A=(A_1,..., A_n) \in P$,

(a) $(\partial F/\partial A_i)\,(A, \cdot) \in C[0, 1]$, $i=1,..., n$.

(b) $F(A, \cdot) \in C[0, 1]$.

Let $\|g\| = \max_{x \in [0,1]} |g(x)|$ for $g \in C[0, 1]$ and let $\|A\|_\infty = \max_{1 \leqslant i \leqslant n} |A_i|$ for $A \in P$. Portions of the proof of the following theorem were enunciated in [1]; but, since in a corollary we use portions of the proof, it will turn out to be convenient to present the entire proof.

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THEOREM 1. *For a given* $f \in C[0, 1]$ *assume that* $F(A^*, \cdot) \in F$ *has the following properties*:

H1. *If* $\{A^{(k)}\} \subset P$ *is any sequence such that* $\{\|f(\cdot) - F(A^{(k)}, \cdot)\| - \|f(\cdot) - F(A^*, \cdot)\| \to 0\}$ *as* $k \to \infty$, *then* $\|A^{(k)} - A^*\|_\infty \to 0$.

H2. $\{(\partial F/\partial A_i)(A^*, \cdot)\}$ *form a weak Tchebycheff system of dimension* n [12].

H3. *There is a set of* $n+1$ *points*, $0 \leqslant x_0 < x_1 < \cdots < x_n \leqslant 1$, *such that*

[1] $|f(x_0) - F(A^*, x_0)| = \|f(\cdot) - F(A^*, \cdot)\|$.

[2] $(f(x_i) - F(A^*, x_i)) = -(f(x_{i+1}) - F(A^*, x_{i+1}))$ $\quad i = 0, 1, \ldots, n-1$.

[3] *For each subset of* n *distinct points* $\{Y_1, \ldots, Y_n\}$ *of the* $n+1$ *points, det* $\{\partial F(A^*, Y_i)/\partial A_j\}_{i,j=1}^n \neq 0$.

Then there is a number $\alpha > 0$ such that for all $A \in P$,

$$\|f(\cdot) - F(A, \cdot)\| \geqslant \|f(\cdot) - F(A^*, \cdot)\| + \alpha \|F(A, \cdot) - F(A^*, \cdot)\|. \quad (1)$$

Proof. Assume the conclusion is not valid. Then there is sequence $\{A^{(k)}\} \subset P$ and a sequence of positive numbers $\alpha_k \to 0$ so that $F(A^{(k)}, \cdot) \not\equiv F(A^*, \cdot)$ for all k and

$$\|f(\cdot) - F(A^{(k)}, \cdot)\| = \|f(\cdot) - F(A^*, \cdot)\| + \alpha_k \|F(A^{(k)}, \cdot) - F(A^*, \cdot)\|. \quad (2)$$

From (2) we obtain

$$\|F(A^*, \cdot) - F(A^{(k)}, \cdot)\| - \|f - F(A^*, \cdot)\| \leqslant \|f - F(A^*, \cdot)\| + \alpha_k \|F(A^*, \cdot) - F(A^{(k)}, \cdot)\|. \quad \text{(2a)}$$

We claim that the sequence $\{\|F(A^{(k)}, \cdot) - F(A^*, \cdot)\|\}$ is bounded. For if this were not so we could find a subsequence (which we do not relabel) which converges to infinity. Dividing equation (2a by $\|F(A^{(k)}) - F(A^*)\|$ and then taking the limit, we reach the contradiction that $\lim_{k \to \infty} \alpha_k \geqslant 1$; hence, the $\{F(A^{(k)}, \cdot)\}$ are bounded in norm. Thus by (2) and H1, $\|A^{(k)} - A^*\|_\infty \to 0$ as $k \to \infty$.

Let $s(x) = \operatorname{sgn}(f(x) - F(A^*, x))$. By H3 and (2) for each x_i,

$$\begin{aligned}\alpha_k \|F(A^{(k)}, \cdot) - F(A^*, \cdot)\| &= \|f(\cdot) - F(A^{(k)}, \cdot)\| - \|f(\cdot) - F(A^*, \cdot)\| \\ &\geqslant s(x_i)(f(x_i) - F(A^{(k)}, x_i)) - s(x_i)(f(x_i) - F(A^*, x_i)) \\ &= s(x_i)(F(A^*, x_i) - F(A^{(k)}, x_i)). \end{aligned} \quad (3)$$

We assert for some $\gamma > 0$,

$$\max_{0 \leqslant i \leqslant n} s(x_i)[F(A^*, x_i) - F(A^{(k)}, x_i)] \geqslant \gamma \cdot \| A^* - A^k \|_\infty \tag{4}$$

for all k. If (4) was never valid there is a subsequence of the $\{A^{(k)}\}$ which we do not relabel and a sequence of positive numbers $\{\gamma_k\}$ such that for all k

$$\max_{0 \leqslant i \leqslant n} s(x_i) \frac{[F(A^*, x_i) - F(A^{(k)}, x_i)]}{\| A^* - A^{(k)} \|_\infty} \leqslant \gamma_k. \tag{5}$$

By the *mean value theorem* for large k,

$$\max_{0 \leqslant i \leqslant n} s(x_i) \left[\sum_{j=1}^{n} \frac{\partial F(A^{(k)}(x_i), x_i)}{\partial A_j} \frac{(-A_j^{(k)} + A_j^*)}{\| A^{(k)} - A^* \|_\infty} \right] \leqslant \gamma_k \tag{6}$$

where $A^{(k)} = (A_1^{(k)}, ..., A_n^{(k)})$ and $A^{(k)}(x_i)$ is on the line between A^* and $A^{(k)}$.

If we set $C^{(k)} = (A^* - A^{(k)})/\| A^* - A^{(k)} \|_\infty$, by going to a subsequence we can assume that $C^{(k)} \to C = (C_1, ..., C_n)$, where $\| C \|_\infty = 1$. Letting $k \to \infty$ in (6) yields

$$\max_{0 \leqslant i \leqslant n} s(x_i) \sum_{j=1}^{n} C_j \frac{\partial F}{\partial A_j} (A^*, x_i) \leqslant 0. \tag{7}$$

From H3 there is a x_l so that $\sum_{j=1}^{n} C_j (\partial F / \partial A_j)(A^*, x_l) \neq 0$ and some element $G(x)$, in the linear span of $\{(\partial F/\partial A_j)(A^*, \cdot)\}_{j=1}^{n}$ with

$$s(x_i) G(x_i) < 0, \qquad i = 0, 1, ..., n; i \neq l.$$

Thus for small positive λ,

$$\sum_{j=1}^{n} C_j \frac{\partial F}{\partial A_j} (A^*, x) + \lambda G(x)$$

has n sign changes, which contradicts the fact that we are dealing with a weak Tchebycheff system of order n [12]. Hence there is a $\gamma > 0$ so that (4) is valid.

Combining (3) and (4) yields

$$\alpha_k \| F(A^{(k)}, \cdot) - F(A^*, \cdot) \| \geqslant \gamma \| A^* - A^{(k)} \|_\infty. \tag{8}$$

By the fact that $A^{(k)} \to A^*$ and by the mean value theorem there is a positive $D > 0$ so that

$$\| F(A^k, \cdot) - F(A^*, \cdot) \| \leqslant D \| A^{(k)} - A^* \|_\infty. \tag{9}$$

Thus from (8) and (9) for large k, $\alpha_k D \geqslant \gamma > 0$, which is a contradiction since $\alpha_k \to 0$. This completes the proof. ∎

The following corollary, as will become clear, is an immediate consequence of the main theorem of [1].

COROLLARY 1. Local Strong Uniqueness Theorem. *Let* $f \in C[0, 1]$ *and* $F(A^*, \cdot)$ *satisfy* H2 *and* H3 *of theorem* 1. *Further assume that for any sequence* $\{F(A^{(k)}, \cdot)\} \subset F$ *such that* $\|F(A^{(k)}, \cdot) - F(A^*, \cdot)\| \to 0$, *we have* $A^{(k)} \to A^*$ *as* $k \to \infty$. *Then there is an* $\varepsilon > 0$ *and a* $\gamma > 0$ *so that*

$$\|F(A^*, \cdot) - F(A, \cdot)\| \leqslant \varepsilon$$
$$\Rightarrow \|f(\cdot) - F(A, \cdot)\| \geqslant \|f(\cdot) - F(A^*, \cdot)\| + \gamma \|F(A, \cdot) - F(A^*, \cdot)\|.$$

Proof. Assume the result is not valid. Then there is a sequence $\{F(A^k)\} \subset F$ and a sequence of positive number $\{\alpha_k\} \to 0$, where $F(A^{(k)}, \cdot) \neq F(A^*, \cdot)$ for all k, $\|F(A^{(k)}, \cdot) - F(A^*, \cdot)\| \to 0$ and (2) of Theorem 1 is valid. By hypothesis $A^{(k)} \to A^*$. The remainder of the proof directly follows the text after Eq. (2) of the cited theorem. ∎

3. APPLICATIONS

Our first application of Theorem 1 is to the problem of securing the polynomial monospline of least uniform norm. Let $n \geqslant 2$ be an integer and m_i an odd positive integer with $n - m_i \geqslant 3$ $(i = 1,..., s)$. Consider the family H of all monosplines of the form,

$$M(x) = \frac{x^n}{n!} + \sum_{j=0}^{n-1} \mathscr{A}_j \Phi_n^{(j)}(x, 0) + \sum_{i=1}^{s} \sum_{j=0}^{m_i - 1} \mathscr{A}_{ij} \Phi_n^{(j)}(x, \xi_i)$$

where the free knots are $0 < \xi_1 < \xi_2 < \cdots < \xi_s < 1$, $\Phi_n^{(j)}(x, \xi) := (\partial^j / \partial \xi^j)$ $\Phi_n(x, \xi)$, and $\Phi_n(x, y)$ is the spline kernel.

It was demonstrated in [2], that there is a unique $M^* \in H$ which minimizes $\|M\| := \max_{x \in [0,1]} |M(x)|$ as $\{\xi_i\}_{i=1}^{s}$, $\{\mathscr{A}_j\}_{j=0}^{n-1}$, and $\{\mathscr{A}_{ij}\}_{j=0, i=1}^{m_i - 1, s}$ vary. Further it was shown that M^* is uniquely characterized by a set of $N + 1$ points $0 = x_0^* < \cdots < x_N^* = 1$ with the properties

$$|M^*(x_0^*)| = \|M^*\|$$
$$M^*(x_i^*) = -M^*(x_{i+1}^*), \qquad i = 0, 1,..., N-1$$

with $N = n + \sum_{i=1}^{s} (m_i + 1)$. Indeed all the hypotheses of Theorem 1 are satisfied except for perhaps H1.

We enlarge H to the set $\bar{H}$, which is defined as follows. For each collection of non-negative integers, $\{r_i\}_{i=0}^{s'}$, $\{m_i'\}_{i=1}^{s'}$, where

$$0 = r_0 < r_1 < \cdots < r_{s'} = s,$$

$$m_i' \leqslant \sum_{i=r_{i-1}+1}^{r_i} m_i \leqslant n-2, \qquad i = 1,..., s'$$

consider any function of the form

$$M(x) = \frac{x^n}{n!} + \sum_{j=0}^{n-1} \mathscr{A}_j \Phi_n^{(j)}(x, 0) + \sum_{i=1}^{s'} \sum_{j=0}^{m_i'-1} \mathscr{A}_{ij} \Phi_n^{(j)}(x, \xi_i).$$

Here $0 \leqslant \xi_i < \xi_{i+1} \leqslant 1$ $(i = 1,..., s'-1)$ {this means the knots ξ_j $j = (r_{i-1}+1),..., r_i$ have coalesced to a single knot ξ_i}. Since H is dense in $\bar{H}$, one has $\min_{\bar{H}} \|M\| = \|M^*\|$. As the following improvement theorem will demonstrate, Hypothesis H1 of theorem 1 is valid.

THEOREM 2. *For* $n \geqslant 2$ *and* $M_1 \in \bar{H} - H$ *there is* $M_2 \in H$ *so that* $\|M_1\| > \|M_2\|$.

Outline of Proof. Using the procedures of [2, 10, 13], it follows that the multiplicities of the knots occurring in an M of minimum norm must be odd. Next we note that if n is odd, and all $m_i' \leqslant n-2$, then no point of alternation can occur at a knot ξ_p of multiplicity $n-2$. For by Theorem 3.2 of [2], any M of minimum norm alternates $N+1$ times, hence $M'(x) = dM(x)/dx$ (which is continuous) has $N-1$ zeros at the interior points of alternation. On the other hand Lemma 3.1 of [2] states that if $M'(x)$ has $N-1$ zeros, and M has a knot of multiplicity $n-2$ at ξ_p then $M'(x)$ cannot vanish at ξ_p. This establishes the result. Multiplicity of knots $n-3$ or less are also easily handled by the procedures of [2, 10, 13]. Hence we need only consider the cases where:

I. $\max_i m_i' = n-1$

II. $\max_i m_i' = n.$

We will demonstrate for these two cases that we can untie the appropriate knots and obtain a smoother $M \in \bar{H}$ whose norm is no larger than the norm of M_1. Further applying the cited results again to the new monospline yields a $M_2 \in H$ whose norm is smaller than that of M_1. We proceed to case I.

We adopt the following notation:

$$\Phi_n(x, y) = \frac{(x-y)_+^{n-1}}{(n-1)!}.$$

The kth order divided difference of the kernel $\Phi_n(x, y)$ at the point x with the knots $y_1, ..., y_k$ will be designated by $\Phi_n[x; y_1, ..., y_k]$. In what follows we will be dealing with the situation where we are taking divided differences using the knots $-\alpha$ and δ. The following symbolisms will be employed:

$$\Phi_n[x; \underbrace{-\alpha, ..., -\alpha}_{k}] := \Phi_n[x; k],$$

$$\Phi_n[x; \underbrace{-\alpha, ..., -\alpha}_{k}, \underbrace{\delta, ..., \delta}_{l}] := \Phi_n[x; k, l].$$

We will designate one-sided limits in the following manner

$$f_+(x_0) = \lim_{x \downarrow x_0} f(x),$$

$$f_-(x_0) = \lim_{x \uparrow x_0} f(x).$$

LEMMA 1. *For $-\alpha \leqslant x \leqslant \delta$, where $\alpha + \delta > 0$,*

$$\Phi_n[x; m, l] = \frac{(\alpha+\delta)^{n-l-m}}{(m-1)!} \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{(-1)^{l+k}}{(n-k-1)!} \times \left(\prod_{v=l}^{l+m-k-2} v \right) \left(\frac{x+\alpha}{\alpha+\delta} \right)^{n-k-1}, \tag{1}$$

where $m + l \leqslant n$. Further if $z = (x+\alpha)/(\alpha+\delta)$,

$$\Phi_n[x; m, l] = (\alpha+\delta)^{n-l-m} p_{m,l}(z), \tag{2}$$

where $p_{m,l}(z)$ is a polynomial of degree $n-1$ with a zero of order $n-m$ at $z = 0$.

Proof. It is easy to show (for example, see [7]) that

$$\Phi_n[x; m, l] = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \frac{\Phi_n(x, z)}{(x-\delta)^l} \bigg|_{z=-\alpha}.$$

The result follows immediately. ∎

Remark. It is well known that for $-\alpha \leqslant x \leqslant \delta$,

$$\Phi_n[x(z); l] = \frac{(-1)^{l-1}}{(l-1)!} (\alpha+\delta)^{n-l} \frac{z^{n-l}}{(n-l)!}, \qquad n > l$$

$$\Phi_n[x; n] = \frac{(-1)^{n-1}}{(n-1)!}, \qquad x \geqslant -\alpha$$

$$= 0, \qquad x < -\alpha.$$

Let $M(x)$ be a continuous polynomial monospline of degree $n \geqslant 3$. Let the origin be an interior point of the interval where we are considering $M(x)$. We assume that at the origin M is not differentiable. Then near the origin the monospline takes the form

$$M(x) = p(x) + \sum_{j=0}^{n-2} b_j \Phi_n[x; \underbrace{0, \ldots, 0}_{j+1}],$$

where p is a polynomial of exact degree n and near "0" $M \in C^0$.

Given an odd integer m which is less that $n-1$, our main problem is to find a monospline with the properties: it agrees with M outside of a small neighborhood of the origin; its uniform norm is no larger than that of M; and finally in this neighborhood it has exactly two knots, one of multiplicity m and the other of multiplicity $n-m-1$.

Without loss of generality we can assume that the jump in M' is negative and n is even. Note that this implies that $b_{n-2} < 0$.

In our development it is useful to consider monosplines which depend on the parameter α and exhibit the local structure,

$$M(x, \alpha) = p(x) + s(x, \alpha),$$

where

$$s(x, \alpha) = \sum_{j=0}^{m-1} b_j(\alpha) \Phi_n[x; j+1] + \sum_{j=0}^{n-m-2} b_{m+j}(\alpha) \Phi_n[x; m, j+1]$$

with $\delta := \delta(\alpha)$. Note that with $b_j(0) = b_j$ $(j = 0, 1, \ldots, n-2)$ and $\alpha = 0 = \delta$, $M(x, 0) = M(x)$.

For $\alpha \geqslant 0$ consider the system of n equations in the n unknowns $(b_0, \ldots, b_{n-2}, \delta)$:

$$\left.\frac{d^i}{dx^i} M_+(x, \alpha)\right|_{x=\delta} = \left.\frac{d^i}{dx^i} M_+(x, 0)\right|_{x=\delta} \qquad (i = 0, 1, \ldots, n-1). \tag{4}$$

We will show first that for small $\alpha \geqslant 0$ there are solutions to the system where $d\delta(\alpha)/d\alpha > 0$. Coupling this with the fact noted above that we have a solution at $\alpha = 0$ will yield a family of monosplines with the correct knot structure.

Equation (4) can be written as

$$f_i(\alpha) := \frac{d^i}{dx^i}\left[s_+(x,\alpha) - s_+(x,0)\right]\Big|_{x=\delta} = 0 \qquad (i = 0, 1,\dots, n-1). \tag{5}$$

In order to use the implicit function theorem we temporarily let $\Phi_n(x,y) = (x-y)^{n-1}/(n-1)!$. Thus once we prove $(d/d\alpha)\,\delta(0) > 0$ and show that the other necessary implicit function conditions are satisfied we are guaranteed valid solutions for $\alpha \geqslant 0$.

The Jacobian matrix J of (5) at ($\alpha = 0 = \delta$; $b_i(0) = b_i$, $i = 0,\dots, n-2$) with

$$J_{ij} = \frac{\partial f_i}{\partial b_j}\begin{Bmatrix} i = 0,\dots, n-1 \\ j = 0,\dots, n-2 \end{Bmatrix}$$

and $J_{i,n-1} = \partial f_i/\partial\delta$ $(i = 0, 1,\dots, n-1)$ is

$$J = \begin{bmatrix} & & & & -(n-m-1)\,b_{n-2}(0) \\ & 0 & \frac{(-1)^{n-2}}{(n-2)!} & & \vdots \\ & & \cdot^{\cdot^{\cdot}} & & \vdots \\ & \frac{1}{2!} & & & \vdots \\ & & & 0 & \vdots \\ -1 & & & & \vdots \\ 1 & & & & \vdots \end{bmatrix}. \tag{6}$$

Since $b_{n-2} = b_{n-2}(0) < 0$, J is non-singular. Further at the initial conditions,

$$\frac{\partial f_0}{\partial \alpha} = m b_{n-2}(0). \tag{7}$$

Invoking the implicit function theorem and employing (6) and (7) and Cramer's rule yields initially:

$$\frac{d\delta(\alpha)}{d\alpha}\Big|_{\alpha=0} = \frac{m}{n-m-1} > 0.$$

Thus we have the desired result; that is, for some c_1, where $0 < c_1 < \frac{1}{2}$ and $0 \leqslant \alpha \leqslant c_1$ implies

> The existence of smooth solutions to (4) with $(d\delta/d\alpha)(\alpha) > 0$ and $b_{n-2}(\alpha) < 0$. (8)

For the moment consider only $x \in [-\alpha, 0)$. Then

$$M(x, \alpha) - M(x, 0) = s(x, \alpha), \tag{9}$$

and with $z = (x + \alpha)/(\alpha + \delta(\alpha))$, (2) and (3) imply that $s(x(z), \alpha)$ has a zero of multiplicity $n - m$ at $z = 0$. Further from (1) and (2)

$$(-1)^n \frac{d^{n-m}}{dz^{n-m}} \Phi_n [x(z); m, n-m-1] \bigg|_{z=0} > 0$$

and hence this derivative is positive. Thus expanding about $z = 0$ and employing (2) and (3),

$$s(x(z), \alpha) = (\alpha + \delta) \left[b_{n-2}(\alpha) \frac{d^{n-m}}{dz^{n-m}} \Phi_n [x(z); m, n-m-1] \, z^{n-m} \right.$$
$$\left. + O(\max\{|\alpha, z^{n-m+1}\}) \right].$$

Thus from (9) there is c_2 with $0 < c_2 < c_1$ so that

$$M(x(z), \alpha) < M(x(z), 0) \tag{10}$$

for $0 \leqslant z \leqslant c_2$ and $0 < \alpha \leqslant c_2$.

For $\alpha > 0$, $M(x, \alpha)$ is of continuity class C^m near δ and for $x = \delta$,

$$\frac{dM_+^{m+1}(x, \alpha)}{dx^{m+1}} - \frac{d^{m+1}}{dx^{m+1}} M_-(x, \alpha) = -b_{n-2}(\alpha)\, \Phi_{n-m-1,-} [x; m, n-m-1]. \tag{10a}$$

Using the characteristic properties of B-splines [6] and the fact that m is odd yields

$$\Phi_{n-m-1,-} [\delta, m, n-m-1] > 0. \tag{11}$$

Thus for $0 < x < \delta$ with the aid of (2), (3), and (4) and the Taylor's series about δ,

$$M(x,\alpha)-M(x,0)=\frac{d^{m+1}}{dx^{m+1}}[M_-(x,\alpha)-M(x,0)]\bigg|_{x=\delta}\frac{(x-\delta)^{m+1}}{(m+1)!}$$
$$+O(|x-\delta|)^{m+2}$$
$$=\frac{d^{m+1}}{dx^{m+1}}[M_-(x,\alpha)-M_+(x,\alpha)]\bigg|_{x=\delta}\frac{(x-\delta)^{m+1}}{(m+1)!}$$
$$+o(|x-\delta|^{m+1}). \tag{12}$$

It follows from (10a), (11), and (12) that there is a c_3 where $0<c_3\leqslant c_2$ so that $0<\alpha\leqslant c_3$ and $|z-1|\leqslant c_3$ imply that

$$M(x(z),\alpha)-M(x(z),0)<0. \tag{13}$$

For $0<\alpha\leqslant c_3$ and $-\alpha\leqslant x\leqslant\delta(\alpha)$, it is clear from (1), (2), and (3) that

$$\frac{d}{dx}s(x,\alpha)=b_{n-2}(\alpha)\,\Phi_{n-1}[x;m,n-m-1]+O[|\alpha|). \tag{14}$$

The function $(d/dx)\,\Phi_{n-1}[x;m,n-m-1]=\Phi_{n-2}[x;m,n-m-1]$ is a B *spline* which is strictly positive in the open interval $(-\alpha,\delta(\alpha))$ [6]. Further $\Phi_{n-1}[-\alpha;m,n-m-1]=0$, $\Phi_{n-1}[\delta(\alpha),m,n-m-1]=1/(n-2)!$ and $\Phi_{n-2}[x(z),m,n-m-1]$ as a function of z is independent of α where again $z(x)=(x+\alpha)/(\alpha+\delta(\alpha))$ for $x\in[-\alpha,\delta(\alpha)]$. In the interval $(-\alpha,0)$,

$$M(x,\alpha)-M(x,0)=s(x,\alpha). \tag{15}$$

In the interval $(0,\delta(\alpha)]$

$$\Phi_{n-1}[x;\underbrace{0,\dots,0}_{j+1}]=\frac{(-1)^j x^{n-1-j}}{(n-1-j)!}=O(|\alpha|) \qquad (j=0,1,\dots,n-3);$$
$$\Phi_{n-1}[x;\underbrace{0,\dots,0}_{n-1}]=\frac{(-1)^{n-2}}{(n-2)!}=\frac{1}{(n-2)!}.$$

Hence

$$\frac{d}{dx}s(x,0)=\frac{b_{n-2}(0)}{(n-2)!}+O(|\alpha|). \tag{16}$$

We note the relationship

$$M(x,\alpha)-M(x,0)=s(x,\alpha)-s(x,0) \qquad \text{for } 0<x\leqslant\delta(\alpha). \tag{17}$$

From (16) there is a positive integer k_0 so that for any integer $k\geqslant k_0$

there is a d_k, where $0<d_k<c_3$ and $d_k \downarrow 0$ so that for $0<\alpha \leqslant d_k$ and $x \in (0, \delta(\alpha))$,

$$\frac{ds}{dx}(x, 0) \leqslant \frac{b_{n-2}(0)}{(n-2)!}+\frac{1}{k}<0.$$

From (14) and the remarks following, for some $\varepsilon>0$ there is a positive $c_4<c_3$ so that $0<\alpha \leqslant c_4$ implies

$$\begin{aligned} &\frac{ds}{dx}(x(z), \alpha)<0 \qquad \text{for } 1 \geqslant z(x) \geqslant c_4 \\ 0>&\frac{ds}{dx}(x(z), \alpha) \geqslant \frac{b_{n-2}(0)}{(n-2)!}+\varepsilon \qquad \text{for } 1-c_4 \geqslant z(x) \geqslant 0. \end{aligned} \tag{18}$$

Hence picking a positive integer k, where $k \geqslant k_0$, $d_k<c_4$ and $1/k<\varepsilon$, we have for $0<\alpha<d_k:=c_5$

$$\frac{d}{dx}[M(x(z), \alpha)-M(x(z), 0)]=\frac{d}{dx} s(x, \alpha)<0$$
$$\text{for } z(x) \geqslant c_5 \text{ and } -\alpha \leqslant x<0, \tag{19}$$

and

$$\begin{aligned} \frac{d}{dx}[M(x(z), \alpha)-M(x(z), 0)] &=\frac{d}{dx} s(x, \alpha)-\frac{d}{dx} s(x, 0) \\ &\geqslant \frac{b_{n-2}(0)}{(n-2)!}+\varepsilon-\frac{b_{n-2}(0)}{(n-2)!}-\frac{1}{k}>0 \end{aligned} \tag{20}$$
$$\text{for } z(x) \leqslant 1-c_5 \text{ and } 0<x<\delta(\alpha).$$

Combining (8), (10), (13), (18), (19), and (20) we obtain

$$M(x, \alpha) \leqslant M(x, 0) \qquad \text{for } x \in[-\alpha, \delta(\alpha)] \text{ and } 0<\alpha<c_5.$$

Letting $c_5>0$ be perhaps even smaller, by continuity the norm of $M(x, \alpha)$ is no larger the norm of $M(x, 0)$ over $[-\alpha, \delta(\alpha)]$. For such α, let

$$M(x, \alpha)=M(x, 0), \qquad x \notin[-\alpha, \delta(\alpha)].$$

Thus we have created a family of monosplines $M(\alpha, x) \in C^m$ with $\|M(\cdot, \alpha)\|_{\infty} \leqslant \|M(\cdot, 0)\|_{\infty}$.

It should be noted that the procedure developed above is a general improvement technique for multiple knots since $p(x)$ can be replaced by any differentiable function. This extends the process for simple knots [5].

Consider "case II," that is, where $M(x)$ is discontinuous. Specifically we assume that the monospline can be written as

$$M(x) = p(x) + \sum_{j=0}^{n-1} b_j \Phi_n[x; \underbrace{0, \ldots, 0}_{j+1}],$$

where $b_{n-1} \neq 0$ and p is a polynomial of exact degree n. Using a technique similar to the one employed for "case I" with $\alpha = \delta$ one can show that for any odd positive integer $m < n-1$ the following is valid: there is a monospline $M_1 \in C^{m-1}$ having near the origin two distinct knots of multiplicities $n-1-m$ and m with $\| M_1 \| \leqslant \| M \|$.[1] ∎

THEOREM 3. *For $n \geqslant 2$ the unique polynomial monospline $M^* \in H$ of minimal norm is strongly unique over H.*

Proof. Our previous results show that any minimizing sequence in H has the property that the knots do not coalesce and moreover the "limit knots" remain in $[0, 1)$. Since the functions which form such monosplines are linearly independent it is clear that the coefficients of the minimizing sequence of monosplines are uniformly bounded. A routine compactness argument based on the uniqueness of M^* demonstrates that the sequence converges uniformly to M^* and indeed the parameters converge to the corresponding parameters of M^*. Thus the hypothesis of *Theorem* 1 is satisfied. ∎

The setting where the kernel $\Phi_n(x, y)$ is replaced by a smooth kernel, $K(x, y)$, which is extended totally positive [12] and the corresponding monosplines contains the constant function "one" (that is, each monospline has a fixed knot y, where $K(x, y) \equiv C > 0$) can be dealt with to prove the *Strong Uniqueness Theorem*. The key ingredient in the proof is the uniqueness result for the monospline of least uniform norm [3]. The actual proof mirrors the proof for polynomial monosplines but is far less intricate because of total positivity and smoothness properties of $K(x, y)$.

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